EE211: Robotic Perception and Intelligence

Lecture 2 Trajectory Generation

Jiankun WANG

Department of Electronic and Electrical Engineering
Southern University of Science and Technology

Undergraduate Course, Sep 2024

Outline

1. Definitions
2. Point-to-Point Trajectories
3. Polynomial Via Point Trajectories

Outline

1 Definitions

2 Point-to-Point Trajectories

3 Polynomial Via Point Trajectories

Background

- Motion planning is the problem of finding a robot motion from a start state to a goal state that avoids obstacles in the environment and satisfies other constraints, such as joint limits or torque limits.
- Path planning is a subproblem of the general motion planning problem. It is a purely geometric problem of finding a collision-free path, without concern for the other constraints.
- Trajectory planning? The feasible path returned by the path planner can be time scaled to create a feasible trajectory.

Modern Robotics.

Definition of Trajectory

- Specification of the robot position as a function of time.
- In detail: Combination of a path, a purely geometric description of the sequence of configurations achieved by the robot, and a time scaling, which specifies the times when those configurations are reached.
- Additional constraints: Respect any given limits on joint velocities, accelerations, or torques.
- Avoid obstacles: Next lecture.

Modern Robotics.

Definition of Trajectory

- A path $\theta(s)$ maps a scalar path parameter s, $\theta : [0, 1] \to \Theta$.
- Sometimes s can be taken to be time and varies from $s = 0$ to the total motion time $s = T$, but it is often useful to separate the role of the geometric path parameter s from the time parameter t.
- A time scaling $s(t)$ assigns a value s to each time $t \in [0, T], \; s : [0, T] \to [0, 1]$.
- A path and a time scaling define a trajectory $\theta(s(t))$ or $\theta(t)$.
- The velocity and acceleration along the trajectory

$$\dot{\theta} = \frac{d\theta}{ds}\dot{s}, \quad \ddot{\theta} = \frac{d\theta}{ds}\ddot{s} + \frac{d^2\theta}{ds^2}\dot{s}^2.$$

Exercises

- Consider a trajectory $\theta = \{x, y, z\}$ is given by $x = \cos 2\pi s$, $y = \sin 2\pi s$, $z = 2s$, $s \in [0, 1]$, and its time scaling is $s(t) = \frac{1}{4}t + \frac{1}{8}t^2$, $t \in [0, 2]$. Calculate $\dot{\theta}$.

Outline

1 Definitions

2 Point-to-Point Trajectories

3 Polynomial Via Point Trajectories

Straight-Line Paths

- Could be defined in joint space or in task space.
- Simplicity in joint space: joint limits typically take the form $\theta_{i,\min} \leq \theta_i \leq \theta_{i,\max}$, the allowable joint configurations form a convex set Θ_{free} in joint space.
- The straight line can be written

$$\theta(s) = \theta_{start} + s(\theta_{end} - \theta_{start}), \ s \in [0, 1].$$

- The derivatives

$$\frac{d\theta}{ds} = \theta_{end} - \theta_{start}, \ \frac{d^2\theta}{ds^2} = 0.$$

Straight-Line Paths

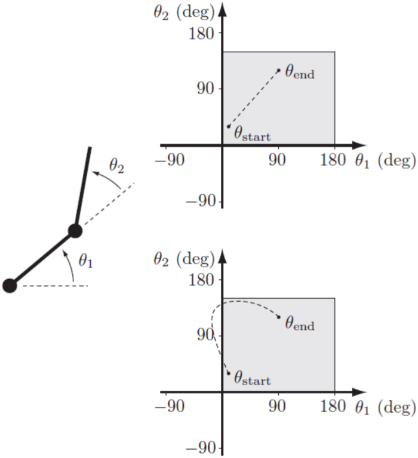


Figure 9.1: (Left) A 2R robot with joint limits $0° \leq \theta_1 \leq 180°$, $0° \leq \theta_2 \leq 150°$. (Top center) A straight-line path in joint space and (top right) the corresponding motion of the end-effector in task space (dashed line). The reachable endpoint configurations, subject to joint limits, are indicated in gray. (Bottom center) This curved line in joint space and (bottom right) the corresponding straight-line path in task space (dashed line) would violate the joint limits.

Time Scaling a Straight-Line Path

- A time scaling $s(t)$ of a path should ensure that the motion is appropriately smooth and that any constraints on robot velocity and acceleration are satisfied.
- Cubic polynomial of time for time scaling $s(t)$

$$s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

- Initial and terminal constraints:

$$s(0) = \dot{s}(0) = 0,\ s(T) = 1, \dot{s}(T) = 0.$$

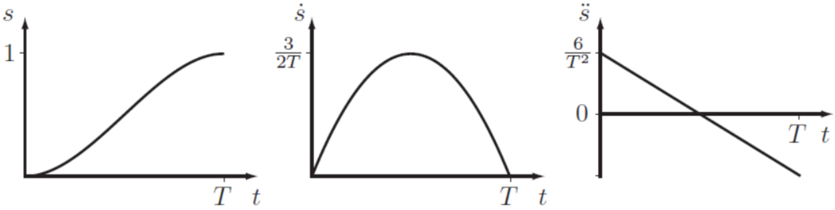
- With the derivative $\dot{s} = a_1 + 2a_2 t + 3a_3 t^2$, we get

$$a_0 = 0, a_1 = 0, a_2 = \frac{3}{T^2}, a_3 = -\frac{2}{T^3}.$$

Time Scaling a Straight-Line Path

- A time scaling $s(t)$ of a path should ensure that the motion is appropriately smooth and that any constraints on robot velocity and acceleration are satisfied.
- Cubic polynomial of time for time scaling $s(t)$

$$s(t) = \frac{3}{T^2}t^2 - \frac{2}{T^3}t^3.$$



Time Scaling a Straight-Line Path

- Substituting $s = a_2 t^2 + a_3 t^3$ into $\theta(s) = \theta_{start} + s(\theta_{end} - \theta_{start})$ yields

$$\theta(t) = \theta_{start} + (\frac{3t^2}{T^2} - \frac{2t^3}{T^3})(\theta_{end} - \theta_{start}),$$
$$\dot{\theta}(t) = (\frac{6t}{T^2} - \frac{6t^2}{T^3})(\theta_{end} - \theta_{start}),$$
$$\ddot{\theta(t)} = (\frac{6}{T^2} - \frac{12t}{T^3})(\theta_{end} - \theta_{start}).$$

- The maximum joint velocities are achieved at the halfway point of the motion, $t = T/2$:

$$\dot{\theta}_{max} = \frac{3}{2T}(\theta_{end} - \theta_{start}).$$

- The maximum joint accelerations and decelerations are achieved at $t = 0$ and $t = T$:

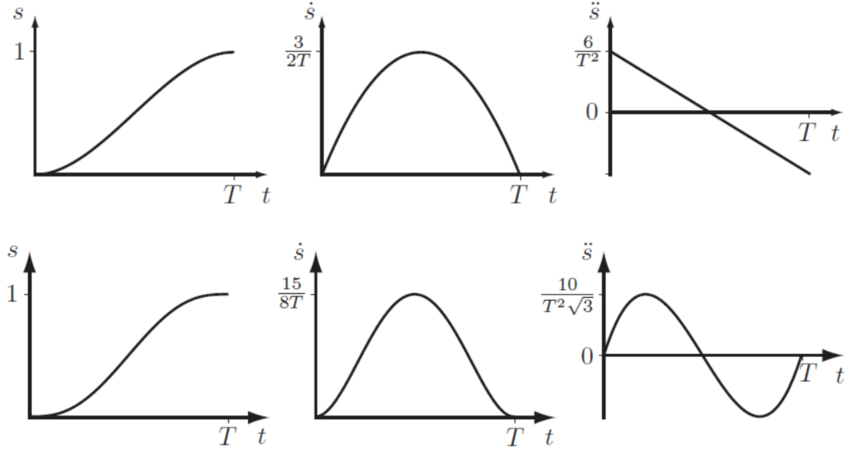
$$\ddot{\theta}_{max} = \frac{6}{T^2}(\theta_{end} - \theta_{start}),\ \ddot{\theta}_{min} = -\frac{6}{T^2}(\theta_{end} - \theta_{start}).$$

Fifth-order Polynomials

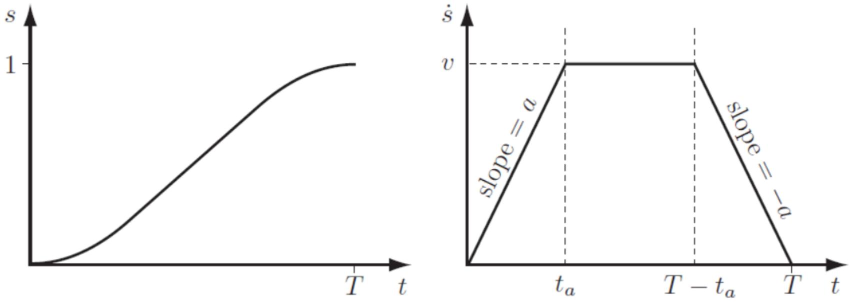
- Third-order time scaling does not constrain the endpoint path accelerations $\ddot{s}(0)$ and $\ddot{s}(T)$ to be zero, the robot is asked to achieve a discontinuous jump in acceleration at both $t = 0$ and $t = T$.
- Constrain start point and endpoint accelerations to $\ddot{s}(0) = \ddot{s}(T) = 0$.
- Requires the addition of two more design freedoms in the polynomial, yielding
$$s(t) = a_0 + \cdots + a_5 t^5.$$
- Use the six terminal position, velocity, and acceleration constraints to solve uniquely for $a_0, \ldots, a_5$.

Fifth-order Polynomials

- Third-order vs. fifth-order profiles



Trapezoidal Motion Profiles

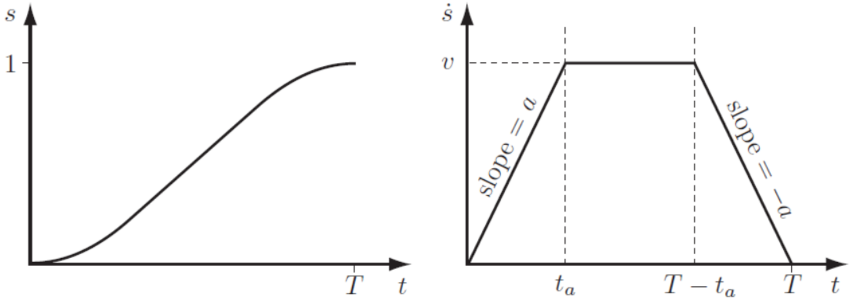


- A constant acceleration phase $\ddot{s} = a$ of time t_a, followed by a constant velocity phase $\dot{s} = v$ of time $t_v = T - 2t_a$, followed by a constant deceleration phase $\ddot{s} = -a$ of time t_a.
- The resulting $\dot{s}$ profile is a trapezoid and the s profile is the concatenation of a parabola, linear segment, and parabola as a function of time.

Trapezoidal Motion Profiles

- Handle constraints on velocities and accelerations.
- Using the largest v and a satisfying $|(\theta_{end} - \theta_{start})v| \leq \dot{\theta}_{limit}$ and $|(\theta_{end} - \theta_{start})a| \leq \ddot{\theta}_{limit}$ is the fastest straight-line motion possible.
- If $v^2/a > 1$, the robot never reaches the velocity v during the motion. The three-phase accelerate-coast-decelerate motion becomes a two-phase accelerate-decelerate "bang-bang" motion.
- If $v^2/a \leq 1$, the trapezoidal motion is fully specified by v, a, t_a, T, but only two of these can be specified independently since they must satisfy $s(T) = 1, v = at_a$.

- Assuming that $v^2/a \leq 1$, the trapezoidal motion is fully specified by v, a, t_a and T.
- Only two of these can be specified independently since they must satisfy $s(T) = 1$ and $v = at_a$.



Trapezoidal Motion Profiles

- By the substitution $t_a = v/a$, then during 3 stages
- Acceleration $0 \le t \le v/a$

$$\ddot{s}(t) = a,\ \dot{s}(t) = at,\ s(t) = \frac{1}{2}at^2;$$

- Constant $v/a < t \le T - v/a$

$$\ddot{s}(t) = 0,\ \dot{s}(t) = v,\ s(t) = vt - \frac{v^2}{2a};$$

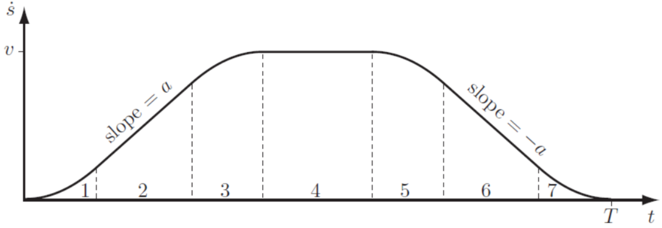
- Deceleration $T - v/a < t \le T$

$$\ddot{s}(t) = -a,\ \dot{s}(t) = a(T - t),\ s(t) = \frac{2avT - 2v^2 - a^2(t - T)^2}{2a}.$$

Trapezoidal Motion Profiles

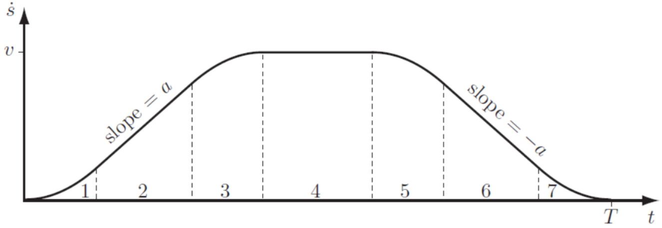
- Only two of v, a, T can be chosen independently, so 3 options:
- Choose v, a such that $v^2/a \leq 1$ and solve $s(T) = 1$, $T = \frac{a+v^2}{va}$. If v, a correspond to the highest possible joint velocities and accelerations, this is the minimum possible time for the motion.
- Choose v, T such that $1 < vT \leq 2$, ensuring a three-stage trapezoidal profile and that the top speed v is sufficient to reach $s = 1$ in time T, and solve $s(T) = 1$, $a = \frac{v^2}{vT-1}$.
- Choose a, T such that $aT^2 \geq 4$, ensuring that the motion is completed in time, and solve $s(T) = 1$, $v = \frac{1}{2}(aT - \sqrt{a}\sqrt{aT^2 - 4})$.

S-Curve Time Scalings

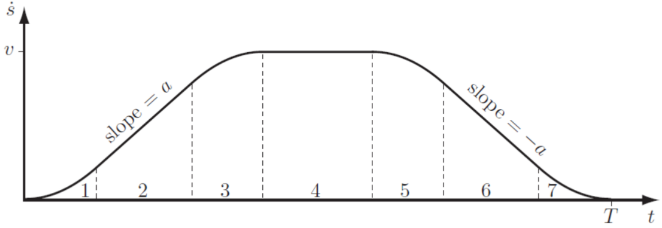


- Trapezoidal motions cause discontinuous jumps in acceleration at $t \in \{0, t_a, T - t_a, T\}$.
- A solution is a smoother S-curve time scaling, a popular motion profile in motor control because it avoids vibrations or oscillations induced by step changes in acceleration.

S-Curve Time Scalings



- (1) Constant jerk $d^3s/dt^3 = J$ until $\ddot{s} = a$; (2) Constant acceleration until $\dot{s} = v$; (3) Constant negative jerk $-J$ until $\ddot{s} = 0$ at the time $\dot{s} = v$; (4) Coasting at constant v; (5) Constant negative jerk $-J$; (6) Constant deceleration $-a$; (7) Constant positive jerk J until $\ddot{s} = \dot{s} = 0$ at the time s reaches 1.



- Time t_j: Duration of a constant positive or negative jerk, including t_1, t_3, t_5, t_7
- Time t_a: Duration of constant positive or negative acceleration, including t_2, t_6
- Time t_v: Duration of constant velocity, including t_4
- Total time T, the jerk J, the acceleration a, and the velocity v.

Exercises

- If you want to use a polynomial time scaling for point-to-point motion with zero initial and final velocities, accelerations, and jerks, what would be the minimum order of the polynomial?

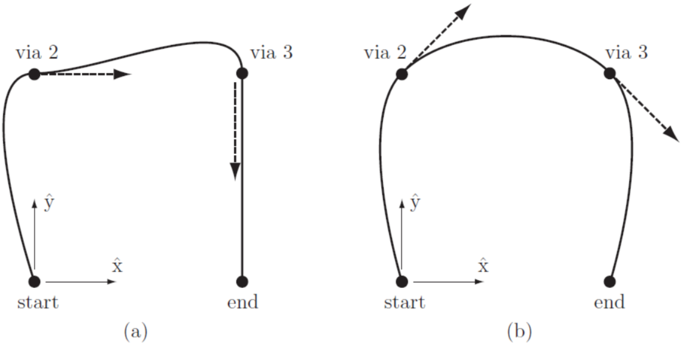
Outline

1 Definitions

2 Point-to-Point Trajectories

3 Polynomial Via Point Trajectories

Polynomial Interpolation



- Pass through a series of via points at specified times, without a strict specification on the shape of path between consecutive points.
- Use histories $\theta(t)$ directly without first specifying a path $\theta(s)$ and then a time scaling $s(t)$.

Polynomial Interpolation

- Specified by k via pints, start at $T_1 = 0$ and end at $T_k = T$.
- At each via point $i \in \{1, ..., k\}$, specified position $\beta(T_i) = \beta_i$ and velocity $\dot{\beta}(T_i) = \dot{\beta}$.
- $k-1$ segments, the duration of segment $j \in \{1, ..., k-1\}$ is $\Delta T_j = T_{j+1} - T_j$, with third-order polynomial, in terms of the time t elapsed in segment j, $0 \leq \Delta t \leq \Delta T_j$,

$$\beta(T_j + \Delta t) = a_{j0} + a_{j1}\Delta t + a_{j2}\Delta t^2 + a_{j3}\Delta t^3.$$

- Segment j is subject to four constraints

$$\beta(T_j) = \beta_j, \ \dot{\beta}(T_j) = \dot{\beta}_j,$$
$$\beta(T_j + \Delta T_j) = \beta_{j+1}, \ \dot{\beta}(T_j + \Delta T_j) = \dot{\beta}_{j+1}.$$

Polynomial Interpolation

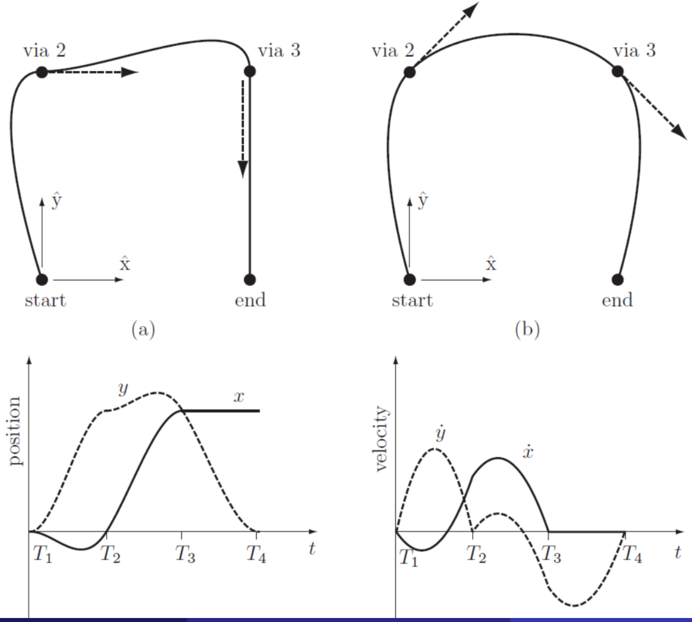
$$\beta(T_j + \Delta t) = a_{j0} + a_{j1}\Delta t + a_{j2}\Delta t^2 + a_{j3}\Delta t^3.$$

$$\beta(T_j) = \beta_j,\ \dot{\beta}(T_j) = \dot{\beta}_j,$$
$$\beta(T_j + \Delta T_j) = \beta_{j+1},\ \dot{\beta}(T_j + \Delta T_j) = \dot{\beta}_{j+1}.$$

- Solving these constraints for $a_{j0}, ..., a_{j3}$ yields

$$a_{j0} = \beta_j,$$
$$a_{j1} = \dot{\beta}_j,$$
$$a_{j2} = \frac{3\beta_{j+1} - 3\beta_j - 2\dot{\beta}_j\Delta T_j - \dot{\beta}_{j+1}\Delta T_j}{\Delta T_j^2},$$
$$a_{j3} = \frac{2\beta_j + (\dot{\beta}_j + \dot{\beta}_{j+1})\Delta T_j - 2\beta_{j+1}}{\Delta T_j^3}.$$

Polynomial Interpolation



Polynomial Interpolation

- The quality of the interpolated trajectories is improved by "reasonable" combinations of via-point times and via-point velocities. A heuristic could be used to choose a via velocity on the basis of the times and coordinate vectors to the via points before and after the via in question.
- Fifth-order polynomials and specification of the accelerations at the via points, at the cost of increased complexity of the solution.
- B-spline interpolation, not pass exactly through the via points, but the path is guaranteed to be confined to the convex hull of the via points.

Exercises

- Via points with specified positions, velocities, and accelerations can be interpolated using fifth-order polynomials of time. For a fifth-order polynomial segment between via points j and $j+1$, of duration ΔT_j, with β_j, β_{j+1}, $\dot{\beta}_j$, $\dot{\beta}_{j+1}$. $\ddot{\beta}_j$, $\ddot{\beta}_{j+1}$ specified, solve for the coefficients of the fifth-order polynomial.

Q & A

1. Definitions
2. Point-to-Point Trajectories
3. Polynomial Via Point Trajectories