EE211: Robotic Perception and Intelligence

Lecture 3 Motion Planning

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Outline

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Formulating The Motion Planning Problem

The Piano Mover's Problem

Given

- A world $\mathcal{W}$, where $\mathcal{W} = \mathbb{R}^2$ or $\mathcal{W} = \mathbb{R}^3$.
- A semi-algebraic obstacle region $\mathcal{O} \subset \mathcal{W}$ in the world.
- A semi-algebraic robot $\mathcal{R}$ in $\mathcal{W}$. It may be a rigid robot or a collection of m rigid elements (links).
- The configuration space $\mathcal{C}$ determined by specifying the set of all possible transformations that can be applied to the robot. Then $\mathcal{C}_{obs}$ and $\mathcal{C}_{free}$ are derived.
- A configuration $q_I \in \mathcal{X}_{free}$ designated as initial configuration.
- A configuration $q_G \in \mathcal{X}_{free}$ designated as goal configuration.

A complete algorithm must compute a continuous path $\tau : [0, 1] \to \mathcal{C}_{free}$, such that $\tau(0) = q_I$ and $\tau(1) = q_G$, or correctly report that such path does not exist.

It has been shown that this problem is PSPACE-hard, which implies NP-hard. Hence, unless P=NP, it is a very hard problem to solve.

Path Planning versus Motion Planning

- Subproblem of the general motion planning problem.
- Path planning is the purely geometric problem, without concern for the dynamics, the duration of motion, or constraints on the motion or on the control inputs.
- The path can be time scaled to create a feasible trajectory.

Other Concepts

- Control inputs: $m = n$ versus $m < n$. Incapable of following any path, even if they are collision-free. A car has $n = 3$ (the position and orientation of the chassis in the plane) but $m = 2$ (forward-backward motion and steering).
- Online versus offline: A motion planning problem requiring an immediate result, because obstacles appear, disappear, or move unpredictably, calls for a fast, online, planner.
- Optimal versus feasible: Objective function $J = \int_0^T L(x(t), u(t))dt$.
- Exact versus approximate: Final state $||x(T) - x_{goal}|| < \epsilon$.
- With or without obstacles: Can be challenging even in the absence of obstacles, particularly if $m < n$ or optimality is desired.

Properties of Motion Planners

- Multiple-query versus single-query planning: Type of problems.
- Anytime planning: Continue to look for better solutions after a first solution is found, until satisfying certain requirements.
- Completeness: A motion planner is said to be complete if it is guaranteed to find a solution in finite time if one exists, and to report failure if there is no feasible motion plan.
- Computational complexity: Characterizations of the amount of time the planner takes to run or the amount of memory it requires.

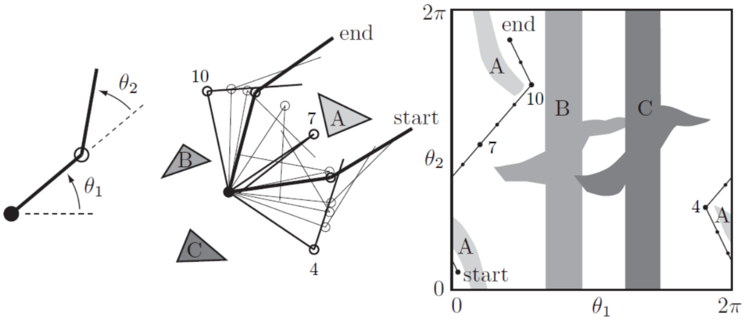
Motion Planning Methods

- Complete methods: Exact representations of the geometry of $\mathcal{C}_{free}$.
- Grid methods: Discretize $\mathcal{C}_{free}$ into a grid.
- Sampling methods: A random or deterministic function to choose a sample from the C-space or state space.
- Virtual potential fields: Online implementation to avoid obstacles.
- Nonlinear optimization: Require an initial guess at the solution, then represent the trajectory or controls by a finite number of design parameters, such as the coefficients of a polynomial.
- Smoothing: Postprocessing to improve the smoothness.

Outline

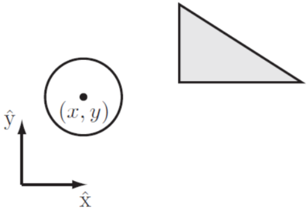
Configuration Space Obstacles

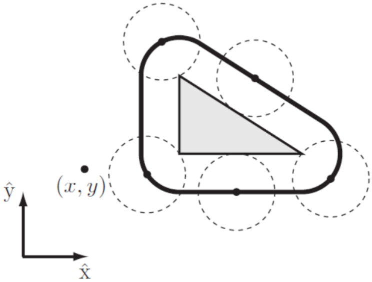
- Degree of C-space equals the degree of freedom of robot.
- Determining whether a robot at a configuration q is in collision with a known environment generally requires a complex operation involving a CAD model of the environment and robot.
- $\mathcal{C} = \mathcal{C}_{free} \cup \mathcal{C}_{obs}$.
- The explicit mathematical representation of a C-obstacle can be exceedingly complex, and for that reason C-obstacles are rarely represented exactly.



- The joint angles of a 2R robot arm.
- The arm navigating among obstacles A, B, and C.
- The same motion in C-space. Three intermediate points, 4, 7, and 10, along the path are labeled.

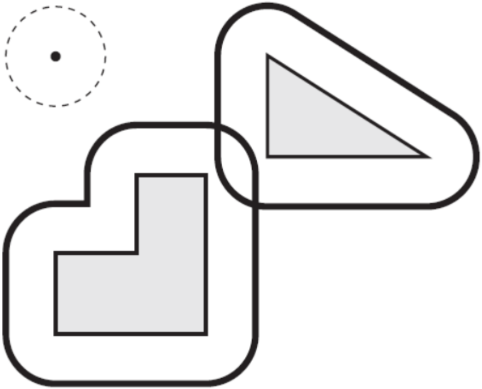
A Circular Planar Mobile Robot





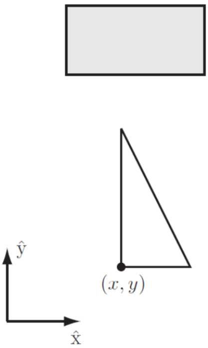
- A circular mobile robot (open circle) and a workspace obstacle (gray triangle). The configuration of the robot is represented by (x, y), the center of the robot.
- In the C-space, the obstacle is "grown" by the radius of the robot and the robot is treated as a point. Any (x, y) configuration outside the bold line is collision-free.

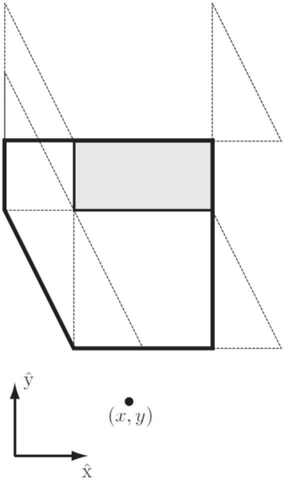
A Circular Planar Mobile Robot



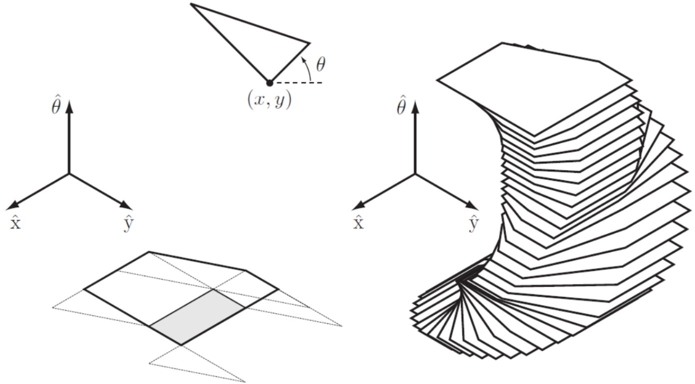
- The “grown” C-space obstacles corresponding to two workspace obstacles and a circular mobile robot. The overlapping boundaries mean that the robot cannot move between the two obstacles.

A Planar Mobile Robot That Translates and Rotates





A Planar Mobile Robot That Translates and Rotates



- A triangular mobile robot that can both rotate and translate, represented by the configuration (x, y, θ).
- The C-space obstacle when the robot is restricted to $\theta = 0$.
- Full 3-dimensional C-space obstacle shown in slices at 10° increments.

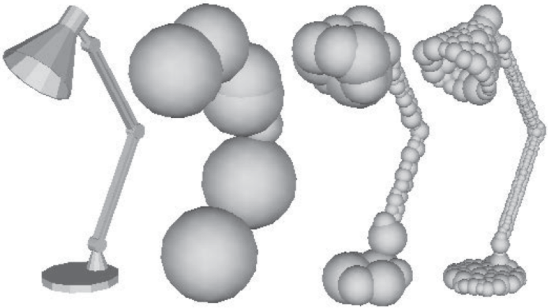
Distance to Obstacles and Collision Detection

- Given a C-obstacle $\mathcal{B}$ and a configuration q, let $d(q, \mathcal{B})$ be the distance between robot and obstacle, then
  - $d(q, \mathcal{B}) > 0$, no contact with the obstacle;
  - $d(q, \mathcal{B}) = 0$, contact;
  - $d(q, \mathcal{B}) < 0$, penetration.
- A distance-measurement algorithm determines $d(q, \mathcal{B})$.
- A collision-detection routine determines whether $d(q, \mathcal{B}_i) \leq 0$ for any C-obstacle $\mathcal{B}_i$.
- A collision-detection routine returns a binary result and may or may not utilize a distance-measurement algorithm at its core.

Distance to Obstacles and Collision Detection

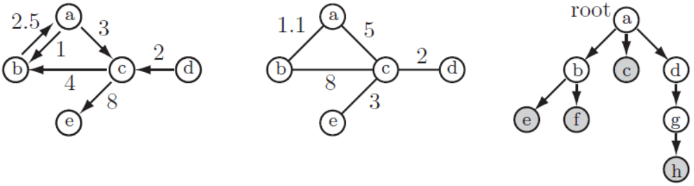
- Approximate the robot and obstacles as unions of overlapping spheres.
- Given a robot at q represented by k spheres of radius R_i centered at $r_i(q)$, $i = 1, ..., k$, and an obstacle $\mathcal{B}$ represented by l spheres of radius $\mathcal{B}_j$ centered at b_j, $j = 1, ..., l$, the distance

$$d(q, \mathcal{B}) = \min_{i,j} ||r_i(q) - b_j|| - R_i - B_j.$$



Graphs and Trees

- Many motion planners explicitly or implicitly represent the C-space or state space as a graph.
- Node $\rightarrow$ configuration or state; edge $\rightarrow$ the ability to move from n_1 to n_2 while satisfying all constraints.
- Directed or undirected.
- Weighted or unweighted.
- A tree is a digraph: (1) no cycles; (2) each node has at most one parent node.



Graphs Search

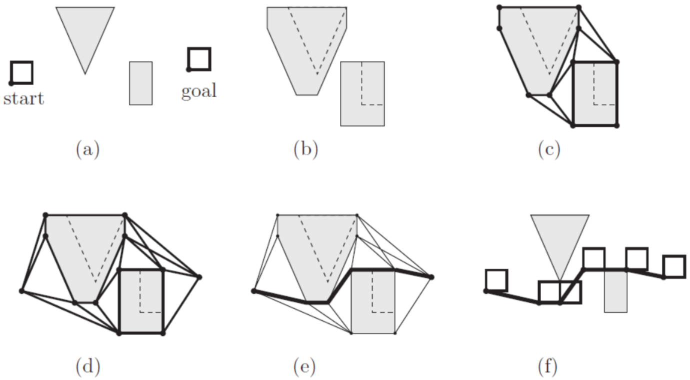
- Once the free space is represented as a graph, a motion plan can be found by searching the graph for a path from the start to the goal.
- Breadth-first search (BFS).
- Depth-first search (DFS).
- Uniform-cost search (UCS).
- Greedy search.
- A* search.

Outline

Complete Path Planners

- Complete path planners rely on an exact representation of the free C-space $\mathcal{C}_{free}$. Mathematically and algorithmically sophisticated, impractical for many real systems.
- Reachability: From every point $q \in \mathcal{C}_{free}$, a free path to a point $q' \in R$ can be found trivially (e.g., a straight-line path).
- Connectivity: For each connected component of $\mathcal{C}_{free}$, there is one connected component of Roadmap R.
- While constructing a roadmap of $\mathcal{C}_{free}$ is complex in general, some problems admit simple roadmaps.
- A suitable roadmap is the weighted undirected visibility graph, with nodes at the vertices of the C-obstacles and edges between the nodes that can "see" each other.

Complete Path Planners

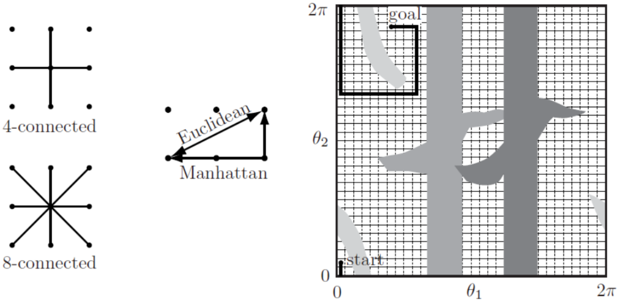


Outline

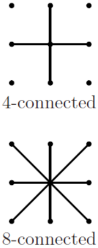
Introduction

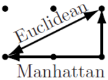
- A search algorithm like A* requires a discretization of the search space. The simplest discretization of C-space is a grid.
- For n-dimensional configuration space, we desire k grid points along each dimension, the C-space is represented by k^n grid points.
- Move in axis-aligned directions or multiple dimensions simultaneously?
- If only axis-aligned motions $\rightarrow$ Manhattan distance.
- A grid-based path planner is resolution complete: it will find a solution if one exists at the level of discretization of the C-space.

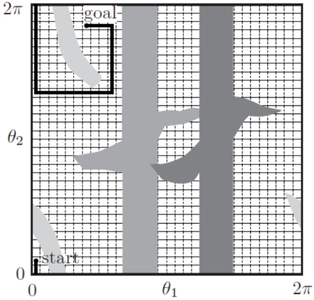
Introduction

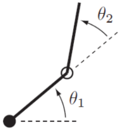


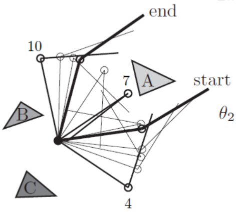
- A 4-connected grid point and an 8-connected grid point for a space $n = 2$.
- Grid points spaced at unit intervals. The Euclidean distance between the two points indicated is $\sqrt{5}$ while the Manhattan distance is 3.
- A grid representation of the C-space and a minimum-length Manhattan-distance path.

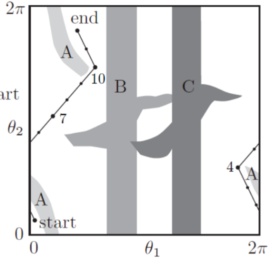

4-connected
8-connected


Euclidean
Manhattan


2π
goal
θ_2
start
0
0
θ_1
2π


θ_2
θ_1


end
10
7
A
B
C
start
4


2π
end
A
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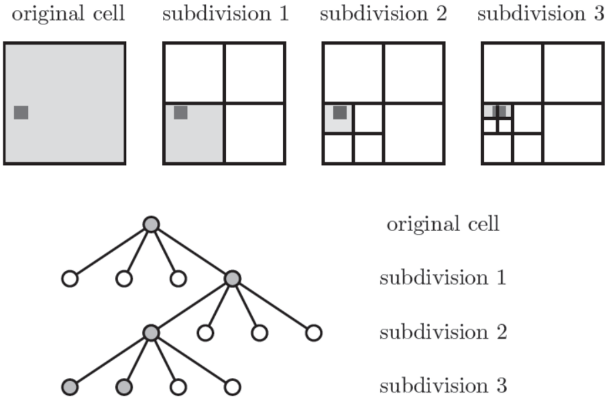
C
7
θ_2
4
A
A
start
0
0
θ_1
2π

Multi-Resolution Grid Representation

- Any cells that are in contact with a C-obstacle are subdivided further, up to a specified maximum resolution.
- Use only 10 cells to represent an obstacle at the same resolution as a fixed grid that uses 64 cells.



Outline

Introduction

- Inspired by potential energy fields in nature, such as gravitational and magnetic fields.
- E.g., the potential energy of a mass m in height h is mgh and the force acting on it is $-mg$. The force will cause the mass to fall to the Earth's surface.
- In robotics, the goal is assigned a low virtual potential and obstacles are assigned a high virtual potential.
- Typically the gradient of the field can be calculated quickly, so the motion can be calculated in real time (reactive control) instead of planned in advance.
- The robot can get stuck in local minima of the potential field, away from the goal, even when a feasible motion to the goal exists.

A Point in C-space

- Assume a point robot in its C-space. q_{goal} is typically encoded by a quadratic potential energy "bowl" with zero energy at the goal

$$\mathcal{P}_{goal}(q) = \frac{1}{2}(q - q_{goal})^T K (q - q_{goal}),$$

where K is a symmetric positive-definite weighting matrix.

- The force induced by this potential is

$$F_{goal}(q) = -\frac{\partial \mathcal{P}_{goal}}{\partial q} = K(q_{goal} - q),$$

an attractive force proportional to the distance from the goal.

A Point in C-space

- The repulsive potential induced by a C-obstacle $\mathcal{B}$ can be calculated from the distance $d(q, \mathcal{B})$ to the obstacle

$$\mathcal{P}_{\mathcal{B}}(q) = \frac{k}{2d^2(q, \mathcal{B})},$$

where $k > 0$ is a scaling factor, $d(q, \mathcal{B}) > 0$.

- The force induced by the obstacle potential is

$$F_{\mathcal{B}}(q) = -\frac{\partial \mathcal{P}_{\mathcal{B}}}{\partial q} = \frac{k}{d^3(q, \mathcal{B})} \frac{\partial d}{\partial q}.$$

A Point in C-space

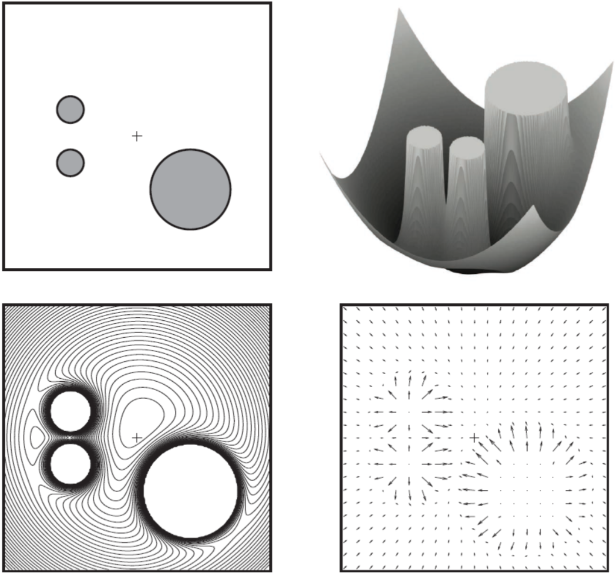
- The total potential is obtained by summing the attractive goal potential and the repulsive obstacle potentials,

$$\mathcal{P}(q) = \mathcal{P}_{goal}(q) + \sum_i \mathcal{P}_{\mathcal{B}_i}(q),$$

yielding a total force

$$F(q) = F_{goal}(q) + \sum_i F_{\mathcal{B}_i}(q).$$

A Point in C-space



A Point in C-space

- To actually control the robot using the calculated $F(q)$, we treat the calculated force as a commanded velocity:

$$u = F(q).$$

- Using the simple obstacle potential, even distant obstacles have a nonzero effect on the motion of the robot.
- To speed up evaluation of the repulsive terms, distant obstacles could be ignored.

$$U_{\mathcal{B}}(q) = \begin{cases} \frac{k}{2}\left(\frac{d_r - d(q,\mathcal{B})}{d_r d(q,\mathcal{B})}\right)^2 & \text{if } d(q,\mathcal{B}) < d_r, \\ 0 & \text{otherwise.} \end{cases}$$

Navigation Functions (Option)

- A significant problem with the potential field method is local minima.
- An approach to local-minimum-free gradient following is based on replacing the virtual potential function with a navigation function.
- A navigation function $\phi(q)$ is a type of virtual potential function that
  - is smooth (or at least twice differentiable) on q;
  - has a bounded maximum value on the boundaries of all obstacles;
  - has a single minimum at q_{goal};
  - has a full-rank Hessian $\partial^2\phi/\partial q^2$ at all critical points q where $\partial\phi/\partial q = 0$.

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Q & A

1. Overview of Motion Planning
2. Foundations
3. Complete Path Planners
4. Grid Methods
5. Virtual Potential Fields